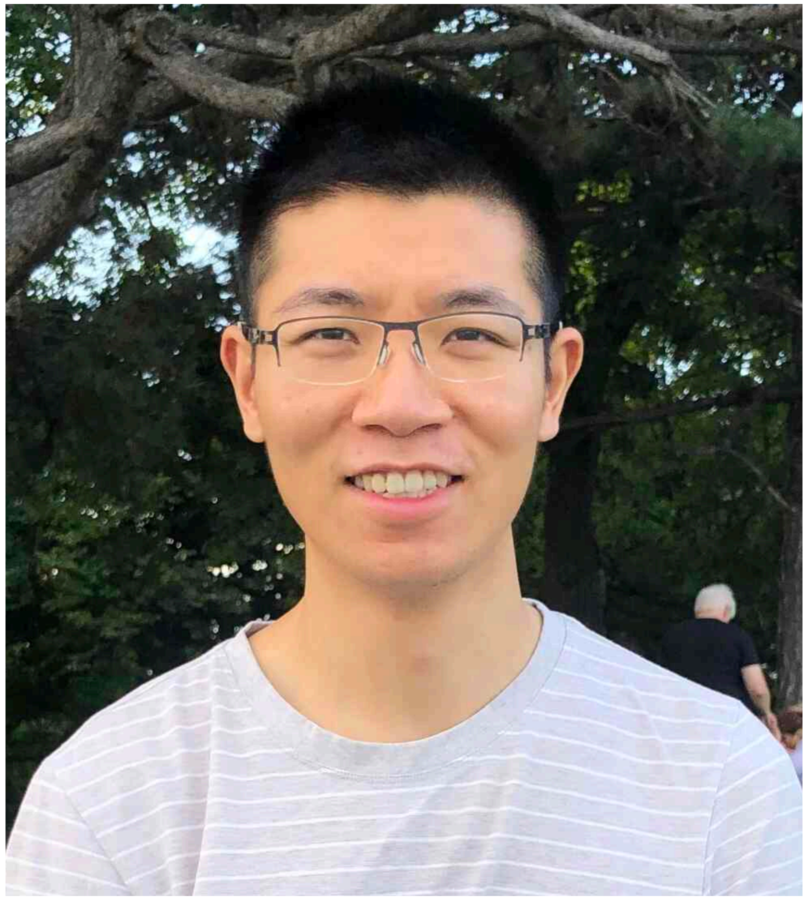
Is Adaptive Robust Confidence Interval Possible?

Chao Gao
University of Chicago

June 2024



Yuetian Luo

Robust CI

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(\theta, 1)$$

Robust CI

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(\theta, 1)$$

$$\mathbb{P}\left(\theta \in \left[\bar{X} \pm \frac{1.96}{\sqrt{n}}\right]\right) \geq 0.95$$

Robust CI

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, 1) + \epsilon Q$$

Robust CI

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, 1) + \epsilon Q$$

optimal estimation

$$\mathbb{P}\left(\left|\widehat{\theta}_{\text{median}} - \theta\right| \leq \sqrt{2\pi}\left(\frac{1.36}{\sqrt{n}} + \epsilon\right)\right) \geq 0.95$$

Robust CI

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, 1) + \epsilon Q$$

optimal estimation

$$\mathbb{P}\left(\left|\widehat{\theta}_{\text{median}} - \theta\right| \leq \sqrt{2\pi}\left(\frac{1.36}{\sqrt{n}} + \epsilon\right)\right) \geq 0.95$$

optimal CI

$$\mathbb{P}\left(\theta \in \left[\widehat{\theta}_{\text{median}} \pm \sqrt{2\pi}\left(\frac{1.36}{\sqrt{n}} + \epsilon\right)\right]\right) \geq 0.95$$

Robust CI

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, 1) + \epsilon Q$$

optimal estimation

$$\mathbb{P}\left(\left|\widehat{\theta}_{\text{median}} - \theta\right| \leq \sqrt{2\pi}\left(\frac{1.36}{\sqrt{n}} + \epsilon\right)\right) \geq 0.95$$

optimal CI

$$\mathbb{P}\left(\theta \in \left[\widehat{\theta}_{\text{median}} \pm \sqrt{2\pi}\left(\frac{1.36}{\sqrt{n}} + \boxed{\epsilon}\right)\right]\right) \geq 0.95$$

known

Adaptive Robust CI

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, 1) + \epsilon Q$$

Adaptive Robust CI

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, 1) + \epsilon Q$$

random interval $\quad \widehat{C} = \widehat{C}(X_1, \cdots, X_n)$

Adaptive Robust CI

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, 1) + \epsilon Q$$

random interval $\widehat{C} = \widehat{C}(X_1, \cdots, X_n)$

coverage $\inf_{\theta, Q} \mathbb{P}_{\epsilon,\theta,Q}\left(\theta \in \widehat{C}\right) \geq 0.95$

Adaptive Robust CI

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, 1) + \epsilon Q$$

random interval $\widehat{C} = \widehat{C}(X_1, \cdots, X_n)$

coverage $$\inf_{\theta,Q} \mathbb{P}_{\epsilon,\theta,Q}\left(\theta \in \widehat{C}\right) \geq 0.95$$

length $$\inf_{\theta,Q} \mathbb{P}_{\epsilon,\theta,Q}\left(\text{length}\left(\widehat{C}\right) \leq r(n,\epsilon)\right) \geq 0.99$$

Adaptive Robust CI

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, 1) + \epsilon Q$$

random interval $\widehat{C} = \widehat{C}(X_1, \cdots, X_n)$

coverage $\inf_{\epsilon \in [0,1/2]} \inf_{\theta, Q} \mathbb{P}_{\epsilon,\theta,Q}\left(\theta \in \widehat{C}\right) \geq 0.95$

length $\inf_{\epsilon \in [0,1/2]} \inf_{\theta, Q} \mathbb{P}_{\epsilon,\theta,Q}\left(\text{length}\left(\widehat{C}\right) \leq r(n, \epsilon)\right) \geq 0.99$

Adaptive Robust CI

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, 1) + \epsilon Q$$

Theorem. For any adaptive CI that satisfies coverage, its length must satisfy

Adaptive Robust CI

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, 1) + \epsilon Q$$

Theorem. For any adaptive CI that satisfies coverage, its length must satisfy

$$r(n, \epsilon) \gtrsim \frac{1}{\sqrt{\log n}} + \frac{1}{\sqrt{\log(1/\epsilon)}}$$

Adaptive Robust CI

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, 1) + \epsilon Q$$

Theorem. For any adaptive CI that satisfies coverage, its length must satisfy

$$r(n, \epsilon) \gtrsim \frac{1}{\sqrt{\log n}} + \frac{1}{\sqrt{\log(1/\epsilon)}}$$

compared with the non-adaptive rate $\frac{1}{\sqrt{n}} + \epsilon$

Adaptive Robust CI

Theorem. With $F_n(t) = \frac{1}{n} \sum_{i \in [n]} \mathbb{I}\{X_i \le t\}$, the CI

Adaptive Robust CI

Theorem. With $F_n(t) = \frac{1}{n}\sum_{i\in[n]} \mathbb{I}\{X_i \le t\}$, the CI

$$\widehat{C} = \left[\max_{t\in[1,\log n]} \left(F_n^{-1}\left(2\left(1-\Phi(t)\right)\right) + t - \frac{1}{t}\right),\right.$$

$$\left.\min_{t\in[1,\log n]} \left(F_n^{-1}\left(1-2\left(1-\Phi(t)\right)\right) - t + \frac{1}{t}\right)\right]$$

Adaptive Robust CI

Theorem. With $F_n(t) = \frac{1}{n}\sum_{i\in[n]} \mathbb{I}\{X_i \le t\}$, the CI

$$\widehat{C} \;=\; \Bigg[\max_{t\in[1,\log n]} \left(F_n^{-1}\left(2\left(1-\Phi(t)\right)\right) + t - \frac{1}{t}\right),$$
$$\min_{t\in[1,\log n]} \left(F_n^{-1}\left(1-2\left(1-\Phi(t)\right)\right) - t + \frac{1}{t}\right)\Bigg]$$

satisfies coverage, and its length satisfies

$$r(n,\epsilon) \lesssim \frac{1}{\sqrt{\log n}} + \frac{1}{\sqrt{\log(1/\epsilon)}}$$

w.h.p.

A Testing Problem

A Testing Problem

A Testing Problem

$$H_0: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(\theta, 1)$$

$$H_1: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2}N(\theta + r, 1) + \frac{1}{2}Q$$

A Testing Problem

$$H_0: \quad X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(\theta, 1)$$

$$H_1: \quad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2}N(\theta + r, 1) + \frac{1}{2}Q$$

$$\inf_{\epsilon \in \{0, 1/2\}} \inf_{\theta, Q} \mathbb{P}_{\epsilon, \theta, Q}\left(\theta \in \widehat{C}\right) \geq 1 - \alpha$$

$$\inf_{\epsilon \in \{0, 1/2\}} \inf_{\theta, Q} \mathbb{P}_{\epsilon, \theta, Q}\left(\text{length}\left(\widehat{C}\right) \leq r(n, \epsilon)\right) \geq 1 - \beta$$

A Testing Problem

$$H_0: \quad X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(\theta, 1)$$

$$H_1: \quad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2}N(\theta + r, 1) + \frac{1}{2}Q$$

$$\phi = \mathbb{I}\{\theta + r \in \widehat{C}(X)\}$$

$$\inf_{\epsilon \in \{0,1/2\}} \inf_{\theta, Q} \mathbb{P}_{\epsilon,\theta,Q}\left(\theta \in \widehat{C}\right) \geq 1 - \alpha$$

$$\inf_{\epsilon \in \{0,1/2\}} \inf_{\theta, Q} \mathbb{P}_{\epsilon,\theta,Q}\left(\text{length}\left(\widehat{C}\right) \leq r(n, \epsilon)\right) \geq 1 - \beta$$

A Testing Problem

$$H_0: \quad X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(\theta, 1)$$

$$H_1: \quad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2} N(\theta + r, 1) + \frac{1}{2} Q$$

$$\widehat{C} = \{\theta + r : \phi_{\theta+r}(X) = 1\}$$

$$\phi = \mathbb{I}\{\theta + r \in \widehat{C}(X)\}$$

$$\inf_{\epsilon \in \{0,1/2\}} \inf_{\theta, Q} \mathbb{P}_{\epsilon,\theta,Q}\left(\theta \in \widehat{C}\right) \geq 1 - \alpha$$

$$\inf_{\epsilon \in \{0,1/2\}} \inf_{\theta, Q} \mathbb{P}_{\epsilon,\theta,Q}\left(\text{length}\left(\widehat{C}\right) \leq r(n, \epsilon)\right) \geq 1 - \beta$$

A Testing Problem

$$H_0 : \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(0,1)$$

$$H_1 : \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2}N(r,1) + \frac{1}{2}Q$$

A Testing Problem

$$H_0: \qquad X_1, \cdots, X_n \stackrel{\text{iid}}{\sim} N(0,1)$$

$$H_1: \qquad X_1, \cdots, X_n \stackrel{\text{iid}}{\sim} \frac{1}{2}N(r,1) + \frac{1}{2}Q$$

$$\mathbb{E}_{H_0}\mathbb{I}\{X \geq t\} = \exp\left(-\frac{1}{2}t^2\right)$$

A Testing Problem

$$H_0: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(0,1)$$

$$H_1: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2}N(r,1) + \frac{1}{2}Q$$

$$\mathbb{E}_{H_0}\mathbb{I}\{X \geq t\} = \exp\left(-\frac{1}{2}t^2\right)$$

$$\mathbb{E}_{H_1}\mathbb{I}\{X \geq t\} \geq \frac{1}{2}\exp\left(-\frac{1}{2}(t-r)^2\right)$$

A Testing Problem

$$H_0: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(0,1)$$

$$H_1: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2} N(r,1) + \frac{1}{2} Q$$

$$\mathbb{E}_{H_0} \mathbb{I}\{X \geq t\} = \exp\left(-\frac{1}{2}t^2\right) \geq \frac{1}{n}$$

$$\mathbb{E}_{H_1} \mathbb{I}\{X \geq t\} \geq \frac{1}{2} \exp\left(-\frac{1}{2}(t-r)^2\right)$$

A Testing Problem

$$H_0: \qquad X_1, \cdots, X_n \stackrel{\text{iid}}{\sim} N(0,1)$$

$$H_1: \qquad X_1, \cdots, X_n \stackrel{\text{iid}}{\sim} \frac{1}{2}N(r,1) + \frac{1}{2}Q$$

$$\mathbb{E}_{H_0}\mathbb{I}\{X \geq t\} = \exp\left(-\frac{1}{2}t^2\right) \geq \frac{1}{n}$$

$$\mathbb{E}_{H_1}\mathbb{I}\{X \geq t\} \geq \frac{1}{2}\exp\left(-\frac{1}{2}(t-r)^2\right) \gg \exp\left(-\frac{1}{2}t^2\right)$$

A Testing Problem

$$H_0: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(0,1)$$

$$H_1: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2}N(r,1) + \frac{1}{2}Q$$

$$\mathbb{E}_{H_0}\mathbb{I}\{X \geq t\} = \exp\left(-\frac{1}{2}t^2\right) \geq \frac{1}{n}$$

$$\mathbb{E}_{H_1}\mathbb{I}\{X \geq t\} \geq \frac{1}{2}\exp\left(-\frac{1}{2}(t-r)^2\right) \gg \exp\left(-\frac{1}{2}t^2\right)$$

Set $t \asymp \sqrt{\log n}$.

A Testing Problem

$$H_0: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(0,1)$$

$$H_1: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2}N(r,1) + \frac{1}{2}Q$$

$$\mathbb{E}_{H_0}\mathbb{I}\{X \geq t\} = \exp\left(-\frac{1}{2}t^2\right) \geq \frac{1}{n}$$

$$\mathbb{E}_{H_1}\mathbb{I}\{X \geq t\} \geq \frac{1}{2}\exp\left(-\frac{1}{2}(t-r)^2\right) \gg \exp\left(-\frac{1}{2}t^2\right)$$

Set $t \asymp \sqrt{\log n}$. When $r \gtrsim \frac{1}{\sqrt{\log n}}$, can separate the two hypotheses.

A Testing Problem

$$H_0: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(0,1)$$

$$H_1: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2}N(r,1) + \frac{1}{2}Q$$

A Testing Problem

$$H_0: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(0,1)$$

$$H_1: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2}N(r,1) + \frac{1}{2}Q$$

$$q(x) \quad \overset{?}{=} \quad 2\phi(x) - \phi(x-r)$$

A Testing Problem

$$H_0: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(0,1)$$

$$H_1: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2}N(r,1) + \frac{1}{2}Q$$

$$q(x) \propto (2\phi(x) - \phi(x-r))\, \mathbb{I}\{|x| \leq t\}$$

A Testing Problem

$$H_0 : \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(0,1)$$

$$H_1 : \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2}N(r,1) + \frac{1}{2}Q$$

$$q(x) \propto (2\phi(x) - \phi(x-r))\,\mathbb{I}\{|x| \leq t\}$$

$$\chi^2\left(\frac{1}{2}N(\theta+r,1) + \frac{1}{2}Q \middle\| N(\theta,1)\right) \ll \frac{1}{n}$$

A Testing Problem

$$H_0: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(0,1)$$

$$H_1: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2}N(r,1) + \frac{1}{2}Q$$

$$q(x) \propto (2\phi(x) - \phi(x-r))\,\mathbb{I}\{|x| \leq t\}$$

$$\chi^2\left(\frac{1}{2}N(\theta+r,1) + \frac{1}{2}Q \,\middle\|\, N(\theta,1)\right) \ll \frac{1}{n} \qquad \exp\left(-\frac{1}{2}x^2\right) \geq \frac{1}{2}\exp\left(-\frac{1}{2}(x-r)^2\right) \text{ for all } |x| \leq t$$

A Testing Problem

$$H_0: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(0,1)$$

$$H_1: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2}N(r,1) + \frac{1}{2}Q$$

$$q(x) \propto (2\phi(x) - \phi(x-r))\,\mathbb{I}\{|x| \leq t\}$$

$$\chi^2\left(\frac{1}{2}N(\theta+r,1) + \frac{1}{2}Q \,\middle\|\, N(\theta,1)\right) \ll \frac{1}{n} \qquad \exp\left(-\frac{1}{2}x^2\right) \geq \frac{1}{2}\exp\left(-\frac{1}{2}(x-r)^2\right) \text{ for all } |x| \leq t$$

Set $t \asymp \sqrt{\log n}$.

A Testing Problem

$$H_0: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(0,1)$$

$$H_1: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2}N(r,1) + \frac{1}{2}Q$$

$$q(x) \propto (2\phi(x) - \phi(x-r))\,\mathbb{I}\{|x| \leq t\}$$

$$\chi^2\left(\frac{1}{2}N(\theta+r,1) + \frac{1}{2}Q \middle\| N(\theta,1)\right) \ll \frac{1}{n} \qquad \exp\left(-\frac{1}{2}x^2\right) \geq \frac{1}{2}\exp\left(-\frac{1}{2}(x-r)^2\right) \text{ for all } |x| \leq t$$

Set $t \asymp \sqrt{\log n}$. When $r \lesssim \frac{1}{\sqrt{\log n}}$, cannot separate the two hypotheses.

A Testing Problem

$$H_0: \qquad X_1, \cdots, X_n \stackrel{\text{iid}}{\sim} N(\theta, 1)$$

$$H_1: \qquad X_1, \cdots, X_n \stackrel{\text{iid}}{\sim} \frac{1}{2}N(\theta + r, 1) + \frac{1}{2}Q$$

A Testing Problem

$$H_0 : \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(\theta, 1)$$

$$H_1 : \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2}N(\theta + r, 1) + \frac{1}{2}Q$$

$$\phi_{\theta+r}(X) = \mathbb{I}\left\{\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}\{X_i > \theta + t\} > 1 - \Phi(t) + \frac{1}{\sqrt{n}}\right\}$$

A Testing Problem

$$H_0: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(\theta, 1)$$

$$H_1: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2}N(\theta + r, 1) + \frac{1}{2}Q$$

$$\phi_{\theta+r}(X) = \mathbb{I}\left\{\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}\{X_i > \theta + t\} > 1 - \Phi(t) + \frac{1}{\sqrt{n}}\right\}$$

$$\{\theta + r : \phi_{\theta+r}(X) = 1\}$$

A Testing Problem

$$H_0: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(\theta, 1)$$

$$H_1: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2} N(\theta + r, 1) + \frac{1}{2} Q$$

$$\phi_{\theta+r}(X) = \mathbb{I}\left\{ \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\{X_i > \theta + t\} > 1 - \Phi(t) + \frac{1}{\sqrt{n}} \right\}$$

$$\{\theta + r : \phi_{\theta+r}(X) = 1\} \quad = \quad \left\{ \theta + r : \theta < F_n^{-1}\left(\Phi(t) - \frac{1}{\sqrt{n}} \right) - t \right\}$$

A Testing Problem

$$H_0: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(\theta, 1)$$

$$H_1: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2} N(\theta + r, 1) + \frac{1}{2} Q$$

$$\phi_{\theta+r}(X) = \mathbb{I}\left\{ \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\{X_i > \theta + t\} > 1 - \Phi(t) + \frac{1}{\sqrt{n}} \right\}$$

$$\begin{aligned} \{\theta + r : \phi_{\theta+r}(X) = 1\} &= \left\{ \theta + r : \theta < F_n^{-1}\left(\Phi(t) - \frac{1}{\sqrt{n}} \right) - t \right\} \\ &= \left\{ \theta : \theta < F_n^{-1}\left(\Phi(t) - \frac{1}{\sqrt{n}} \right) - t + r \right\} \end{aligned}$$

A Testing Problem

$$H_0: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(\theta, 1)$$

$$H_1: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2}N(\theta + r, 1) + \frac{1}{2}Q$$

$$\{\theta + r : \phi_{\theta+r}(X) = 1\} \quad = \quad \left\{\theta : \theta < F_n^{-1}\left(\Phi(t) - \frac{1}{\sqrt{n}}\right) - t + r\right\}$$

A Testing Problem

$$H_0: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, 1) + \epsilon Q$$

$$H_1: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2}N(\theta + r, 1) + \frac{1}{2}Q$$

$$\{\theta + r : \phi_{\theta+r}(X) = 1\} \quad = \quad \left\{\theta : \theta < F_n^{-1}\left(\Phi(t) - \frac{1}{\sqrt{n}} - \epsilon\right) - t + r\right\}$$

A Testing Problem

$$H_0: \quad X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta,1) + \epsilon Q$$

$$H_1: \quad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2}N(\theta+r,1) + \frac{1}{2}Q$$

$$\{\theta + r : \phi_{\theta+r}(X) = 1\} = \left\{\theta : \theta < F_n^{-1}\left(\Phi(t) - \frac{1}{\sqrt{n}} - \epsilon\right) - t + r\right\}$$

$$= \left\{\theta : \theta < F_n^{-1}\left(1 - 2(1-\Phi(t))\right) - t + \frac{1}{t}\right\}$$

A Testing Problem

$$H_0: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta,1) + \epsilon Q$$

$$H_1: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2}N(\theta+r,1) + \frac{1}{2}Q$$

$$\begin{aligned}\{\theta + r : \phi_{\theta+r}(X) = 1\} &= \left\{\theta : \theta < F_n^{-1}\left(\Phi(t) - \frac{1}{\sqrt{n}} - \epsilon\right) - t + r\right\} \\ &= \left\{\theta : \theta < F_n^{-1}\left(1 - 2(1-\Phi(t))\right) - t + \frac{1}{t}\right\}\end{aligned}$$

$$\widehat{C} = \left[\max_{t\in[1,\log n]}\left(F_n^{-1}\left(2\left(1-\Phi(t)\right)\right) + t - \frac{1}{t}\right), \min_{t\in[1,\log n]}\left(F_n^{-1}\left(1 - 2\left(1-\Phi(t)\right)\right) - t + \frac{1}{t}\right)\right]$$

What about other distributions?

Adaptive Robust CI

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)\text{Laplace}(\theta, 1) + \epsilon Q$$

Adaptive Robust CI

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)\text{Laplace}(\theta, 1) + \epsilon Q$$

$$\frac{1}{2}\exp\left(-|x-\theta|\right)$$

Adaptive Robust CI

$$X_1, \cdots, X_n \stackrel{\text{iid}}{\sim} (1-\epsilon)\text{Laplace}(\theta, 1) + \epsilon Q$$

$$\frac{1}{2}\exp\left(-|x-\theta|\right)$$

Theorem. For Laplace distribution, any adaptive CI that satisfies coverage must have length

$$r(n, \epsilon) \gtrsim 1$$

A Testing Problem

$$H_0 : \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} L(0,1)$$

$$H_1 : \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2}L(r,1) + \frac{1}{2}Q$$

A Testing Problem

$$H_0: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} L(0,1)$$

$$H_1: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2}L(r,1) + \frac{1}{2}Q$$

$$q(x) = \exp\left(-|x|\right) - \frac{1}{2}\exp\left(-|x-r|\right) \geq 0 \text{ for all } x \in \mathbb{R}$$

A Testing Problem

$$H_0: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} L(0,1)$$

$$H_1: \qquad X_1, \cdots, X_n \overset{\text{iid}}{\sim} \frac{1}{2} L(r,1) + \frac{1}{2} Q$$

$$q(x) = \exp\left(-|x|\right) - \frac{1}{2}\exp\left(-|x-r|\right) \geq 0 \text{ for all } x \in \mathbb{R}$$

When $r \leq \log 2$, cannot separate the two hypotheses.

Adaptive Robust CI

	density	optimal estimation error	optimal CI length
t-distribution	$(1+x^2)^{-\frac{\nu+1}{2}}$		
generalized Gaussian	$\exp\left(-\lvert x\rvert^{\beta}\right)$		
mollifier	$\exp\left(-\frac{1}{(1-x^2)^{\beta}}\right)\mathbb{I}\{\lvert x\rvert \leq 1\}$		
Bates	$U_1 + \cdots + U_k \quad k \geq 2$		

Adaptive Robust CI

	density	optimal estimation error	optimal CI length
t-distribution	$(1+x^2)^{-\frac{\nu+1}{2}}$	$\frac{1}{\sqrt{n}} + \epsilon$	
generalized Gaussian	$\exp\left(-\lvert x\rvert^{\beta}\right)$	$\frac{1}{\sqrt{n}} + \epsilon$	
mollifier	$\exp\left(-\frac{1}{(1-x^2)^{\beta}}\right)\mathbb{I}\{\lvert x\rvert \le 1\}$	$\frac{1}{\sqrt{n}} + \epsilon$	
Bates	$U_1 + \cdots + U_k \quad k \ge 2$	$\frac{1}{\sqrt{n}} + \epsilon$	

Adaptive Robust CI

	density	optimal estimation error	optimal CI length
t-distribution	$(1+x^2)^{-\frac{\nu+1}{2}}$	$\frac{1}{\sqrt{n}}+\epsilon$	1
generalized Gaussian	$\exp\left(-\lvert x\rvert^{\beta}\right)$	$\frac{1}{\sqrt{n}}+\epsilon$	$\left(\frac{1}{\log n}+\frac{1}{\log(1/\epsilon)}\right)^{\frac{(\beta-1)_+}{\beta}}$
mollifier	$\exp\left(-\frac{1}{(1-x^2)^{\beta}}\right)\mathbb{I}\{\lvert x\rvert\leq 1\}$	$\frac{1}{\sqrt{n}}+\epsilon$	$\left(\frac{1}{\log n}+\frac{1}{\log(1/\epsilon)}\right)^{\frac{\beta+1}{\beta}}$
Bates	$U_1+\cdots+U_k \quad k\geq 2$	$\frac{1}{\sqrt{n}}+\epsilon$	$\left(\frac{1}{n}+\epsilon\right)^{1/k}$

High-Dimensional Confidence Set

Adaptive Robust CS

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, I_p) + \epsilon Q$$

Adaptive Robust CS

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, I_p) + \epsilon Q$$

random set $\quad \widehat{C} = \widehat{C}(X_1, \cdots, X_n)$

Adaptive Robust CS

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, I_p) + \epsilon Q$$

random set $\widehat{C} = \widehat{C}(X_1, \cdots, X_n)$

coverage $\inf_{\epsilon \in [0,1/2]} \inf_{\theta, Q} \mathbb{P}_{\epsilon,\theta,Q}\left(\theta \in \widehat{C}\right) \geq 0.95$

Adaptive Robust CS

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, I_p) + \epsilon Q$$

random set $\quad \widehat{C} = \widehat{C}(X_1, \cdots, X_n)$

coverage $\quad \inf_{\epsilon \in [0,1/2]} \inf_{\theta, Q} \mathbb{P}_{\epsilon,\theta,Q}\left(\theta \in \widehat{C}\right) \geq 0.95$

length $\quad \inf_{\epsilon \in [0,1/2]} \inf_{\theta, Q} \mathbb{P}_{\epsilon,\theta,Q}\left(\text{radius}\left(\widehat{C}\right) \leq r(n,p,\epsilon)\right) \geq 0.99$

Adaptive Robust CS

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, I_p) + \epsilon Q$$

Theorem. For any adaptive CS that satisfies coverage, its length must satisfy

$$r(n,p,\epsilon) \gtrsim \frac{1}{\sqrt{\log(n/p)}} + \frac{1}{\sqrt{\log(1/\epsilon)}}$$

Moreover, there exists an algorithm that achieves the lower bound.

Adaptive Robust CI

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, 1) + \epsilon Q$$

Adaptive Robust CI

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, 1) + \epsilon Q$$

$$\begin{aligned}\widehat{C} \quad = \quad & \left[\max_{t \in [1, \log n]} \left(F_n^{-1}\left(2\left(1 - \Phi(t)\right)\right) + t - \frac{1}{t} \right), \right. \\ & \left. \min_{t \in [1, \log n]} \left(F_n^{-1}\left(1 - 2\left(1 - \Phi(t)\right)\right) - t + \frac{1}{t} \right) \right]\end{aligned}$$

Adaptive Robust CI

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, 1) + \epsilon Q$$

$$\begin{aligned}\widehat{R} &= \min_{t \in [1, \log n]} \left(F_n^{-1}\left(1 - 2\left(1 - \Phi(t)\right)\right) - t + \frac{1}{t} \right) \\ &\quad - \max_{t \in [1, \log n]} \left(F_n^{-1}\left(2\left(1 - \Phi(t)\right)\right) + t - \frac{1}{t} \right)\end{aligned}$$

Adaptive Robust CI

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, 1) + \epsilon Q$$

$$\begin{aligned}\widehat{R} \;&=\; \min_{t \in [1, \log n]} \left(F_n^{-1}\left(1 - 2\left(1 - \Phi(t)\right)\right) - t + \frac{1}{t} \right) \\ &\quad - \max_{t \in [1, \log n]} \left(F_n^{-1}\left(2\left(1 - \Phi(t)\right)\right) + t - \frac{1}{t} \right)\end{aligned}$$

$$\widehat{C} = \left[\widehat{\theta}_{\text{median}} - \widehat{R}, \widehat{\theta}_{\text{median}} + \widehat{R} \right]$$

Adaptive Robust CS

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, I_p) + \epsilon Q$$

Adaptive Robust CS

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, I_p) + \epsilon Q$$

$$\widehat{C} = \left\{\theta : \|\theta - \widehat{\theta}\| \leq \widehat{R}\right\}$$

Adaptive Robust CS

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, I_p) + \epsilon Q$$

$$\widehat{C} = \left\{ \theta : \|\theta - \widehat{\theta}\| \leq \widehat{R} \right\}$$

Tukey's median
LRV or DKKLMS

Adaptive Robust CS

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, I_p) + \epsilon Q$$

$$\widehat{C} = \left\{ \theta : \|\theta - \widehat{\theta}\| \leq \widehat{R} \right\}$$

Tukey's median
LRV or DKKLMS

$$F_n(u,t) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}\{u^T X_i \leq t\}$$

Adaptive Robust CS

$$X_1, \cdots, X_n \overset{\text{iid}}{\sim} (1-\epsilon)N(\theta, I_p) + \epsilon Q$$

$$\widehat{C} = \left\{ \theta : \|\theta - \widehat{\theta}\| \leq \widehat{R} \right\}$$

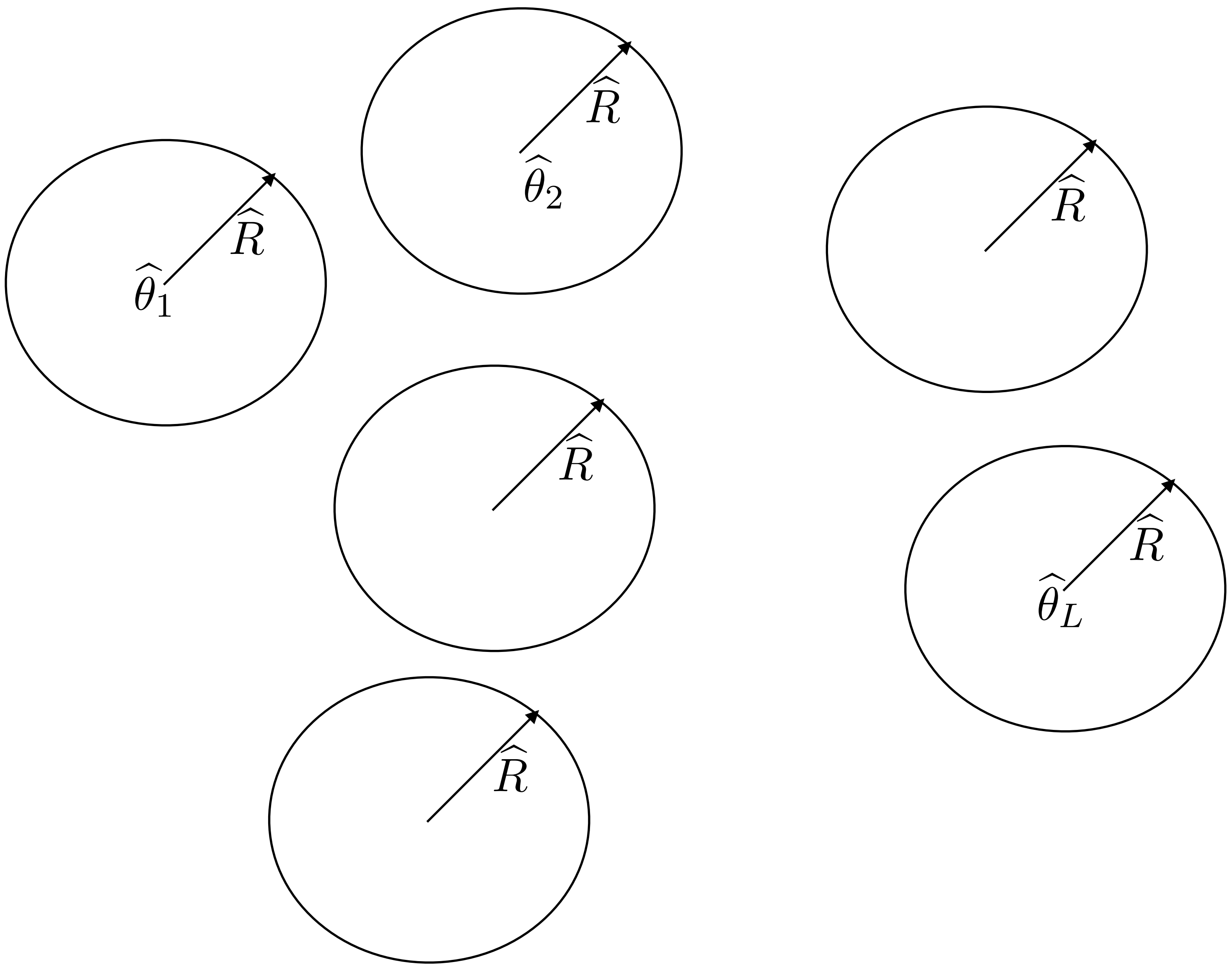
Tukey's median
LRV or DKKLMS

$$F_n(u,t) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}\{u^T X_i \leq t\}$$

$$\begin{aligned} \widehat{R} &= \max_{\|u\|=1} \left[\min_{t \in [1, \log(n/p)]} \left(F_n^{-1}\left(u, 1 - 2\left(1 - \Phi(t)\right)\right) - t + \frac{1}{t} \right) \right. \\ &\quad \left. - \max_{t \in [1, \log(n/p)]} \left(F_n^{-1}\left(u, 2\left(1 - \Phi(t)\right)\right) + t - \frac{1}{t} \right) \right] \end{aligned}$$

Robust CS and List-Decodable Estimation

Adaptive Robust CS



Thank You